Linear Algebra and its Applications 430 (2009) 1436–1440

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Note

A note on the $p \to q$ norms of 2-positive maps

Koenraad M.R. Audenaert

Mathematics Department, Royal Holloway, University of London, Egham TW20 0EX, United Kingdom

ARTICLE INFO

Article history:
Received 30 September 2005
Accepted 29 September 2008
Available online 20 November 2008

Submitted by R.A. Brualdi

Keywords:
2-positive map
Schatten norm
Maximal output purity
Quantum channel

ABSTRACT

King and Ruskai asked whether the $p \to q$ norm of a completely positive map Φ, acting between Schatten p and q classes of Hermitian operators

$$\|\Phi\|_{p\to q} = \sup_{A=A^*} \frac{\|\Phi(A)\|_q}{\|A\|_p},$$

is equal to the $p \to q$ norm of that map when acting between Schatten classes of general, not necessarily Hermitian, operators. The first proof of this statement has been given by Watrous. Here, we give an alternative proof that is also valid for 2-positive maps Φ.

This note is concerned with a mathematical question that is related to certain problems in quantum information theory. In quantum theory, states of finite-dimensional physical systems are represented by density matrices, which are positive semi-definite (PSD) matrices with trace 1 supported on a Hilbert space $\mathscr{H}$ of appropriate dimension; the set of states will be denoted by $\mathscr{S}(\mathscr{H})$, and is a subset of the $*$-algebra $\mathscr{B}(\mathscr{H})$ of operators in $\mathscr{H}$. The action of a physical apparatus on such physical systems can be represented by a completely positive, trace preserving linear map of $\mathscr{B}(\mathscr{H})$, also called *quantum channel*, acting on the state's density matrix [5,6].

It is obvious that physical maps must be positive and trace preserving. A linear map Φ is positive if and only if for all PSD matrices ρ, $\Phi(\rho)$ is also PSD; it is trace preserving if and only if $\mathrm{Tr}\,\Phi(\rho) = \mathrm{Tr}\,\rho$ for all states ρ. The physical reason for the requirement is that physical states must be mapped on physical states. Complete positivity has to do with the fact that maps may also be made to operate on subsystems. A physical system consisting of two subsystems can be represented by a density matrix supported on a tensor product of two Hilbert spaces, $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, where $\mathscr{H}_i$ corresponds to the ith subsystem and has the appropriate dimension; let d be the dimension of $\mathscr{H}_1$, and n the dimension of $\mathscr{H}_2$. The physical significance of the distinction between the two spaces lies in the fact that the two subsystems may reside in different places, potentially separated by large distances, so that for example not all general basis transformations on $\mathscr{H}$ are physically possible.

E-mail address: koenraad.audenaert@rhul.ac.uk

0024-3795/$ - see front matter © 2008 Published by Elsevier Inc.
doi:10.1016/j.laa.2008.09.040

A linear map Φ acting on $\mathscr{H}_1$ is completely positive if and only if the map $\Phi \otimes \mathbb{1}_n$, which acts as Φ on $\mathscr{H}_1$ but leaves subsystem 2 invariant, is positive, for all values of $n \in \mathbb{N}$. Positive maps Φ for which $\Phi \otimes \mathbb{1}_n$ is positive for a given n are called n-positive maps. One can show that a d-dimensional linear map is completely positive if and only if it is d-positive.

A mathematically convenient way to express n-positivity is by using a block matrix notation. Let ρ be a hermitian, PSD block matrix in $\mathbb{M}_n(\mathbb{M}_d(\mathbb{C}))$, $\rho = [A_{i,j}]_{i,j=1}^n$, with $A_{i,j} \in \mathbb{M}_d(\mathbb{C})$, then $(\Phi \otimes \mathbb{1}_n)(\rho)$ is the induced map, represented by the block matrix $[\Phi(A_{i,j})]_{i,j=1}^n$. Thus, for example, a map is 2-positive if and only if

$$\begin{pmatrix} \Phi(A) & \Phi(B) \\ \Phi(B^*) & \Phi(D) \end{pmatrix} \geqslant 0 \quad \forall \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \geqslant 0.$$

Complete positivity can be completely characterised; a linear map Φ is CP if and only if it can be represented as $\Phi(\rho) = \sum_k A_k \rho A_k^*$ for certain operators A_k. Another characterisation has been given by Choi [3]. It is still an open problem to find a corresponding characterisation of n-positivity.

An important property of a quantum state is its *purity*. When a state can be represented by a state vector ψ (the finite dimensional version of a wave function), its density matrix is the projector $\psi\psi^*$ on that vector. Such a state is called a pure state and is represented by a density matrix of rank 1. In general, for a variety of reasons, states have to be represented by statistical mixtures of state vectors, and in that case the density matrix is no longer rank 1. Such states are called mixed states. The maximally mixed state is represented by the scalar matrix $\mathbb{1}_d/d$. Mathematically speaking, the set of states $\mathscr{S}(\mathscr{H})$ forms a compact convex subset of $\mathscr{B}(\mathscr{H})$, with the pure states as extremal points.

The amount of mixing can be expressed by the so-called purity of the quantum state. Several ways to do so exist, and in this note we restrict to purity measures based on the Schatten q-norm. The Schatten q-norm is the non-commutative generalisation of the ℓ_q norms: for a $d \times d$ matrix A and for $q \geqslant 1$

$$\|A\|_q := (\mathrm{Tr}(|A|^q))^{1/q},$$

where the absolute value $|A|$ is defined as

$$|A| = (A^*A)^{1/2}.$$

The operator norm $\|A\|$ is the limiting case of q going to $+\infty$. The maximal value $\|A\|_q$ can attain is 1, for pure states, and the minimal value is $d^{1/q-1}$, attained for the maximally mixed state $\mathbb{1}_d/d$.

A corresponding property of a CP map is its *maximal output purity*. The maximal output purity of a completely positive linear map Φ, as measured by the Schatten q-norm, is defined as

$$\nu_q(\Phi) = \max_{\substack{A \geqslant 0 \\ \mathrm{Tr}(A)=1}} \|\Phi(A)\|_q.$$

I.e. this gives the highest purity (measured by the q-norm) an output state of the channel can attain when varying over all possible input states. By convexity of the norm, the maximum is attained for pure input states. This quantity is a measure of "noisiness" of the map [1], considering it as a channel for the quantum state A. It expresses the fact that a pure state will acquire at least this amount of mixedness when passing through the channel.

In Ref. [1] the following Proposition has been proven:

Proposition 1. *The maximal output q-purity of a positive map Φ is equal to the $1 \to q$ norm of Φ, considered as a map from the Schatten 1-class $\ell_1(\mathscr{H})$ of Hermitian operators to the Schatten q-class $\ell_q(\mathscr{H})$. That is,*

$$\nu_q(\Phi) = \sup_{\substack{A=A^* \\ A \neq 0}} \frac{\|\Phi(A)\|_q}{\|A\|_1}, \tag{1}$$

where the condition of positivity of A is no longer required.

For ease of reference we give a proof below.

Proof. Obviously, $\nu_q(\Phi) \leqslant \|\Phi\|_{1\to q}$, because the latter quantity involves a supremum over a larger set. Conversely, we consider the Jordan decomposition $A = A_+ - A_-$ of A in its positive and negative parts A_+ and A_-, which are both PSD by definition. Then $|A| = A_+ - A_-$. By positivity of Φ, we have $\Phi(A_+) \geqslant 0$, $\Phi(A_-) \geqslant 0$. Now note that $\Phi(A) = \Phi(A_+) - \Phi(A_-)$ and $\Phi(|A|) = \Phi(A_+) + \Phi(A_-)$. We can then use the folk lemma (see, e.g. [2, Lemma 1]) that for any unitarily invariant norm and for $X, Y \geqslant 0$, $\|X - Y\| \leqslant \|X + Y\|$, to conclude that

$$\|\Phi(A)\|_q \leqslant \|\Phi(|A|)\|_q. \tag{2}$$

The statement of the proposition follows immediately by noting that $|A|$ can now take over the role of ρ in the definition of ν_q. □

More generally, one defines the $p \to q$ norm of a CP map acting from $\ell_p(\mathscr{H})$ to $\ell_q(\mathscr{H})$ as

$$\|\Phi\|_{p\to q} := \sup_{A=A^*} \frac{\|\Phi(A)\|_q}{\|A\|_p}.$$

King and Ruskai asked [4] whether the condition that A be Hermitian can also be dropped; that is, whether the following holds:

$$\|\Phi\|_{p\to q} = \sup_{A} \frac{\|\Phi(A)\|_q}{\|A\|_p}.$$

They have proved this in [4] for the case $p = q = 2$, and for the case of trace preserving CP maps between 2-dimensional Schatten classes with $p = 1$ and $q \geqslant 2$.

A general proof has been given by Watrous [8] for any CP map and for all values of $p, q \geqslant 1$. In this note we present an alternative proof, which shows that the statement holds somewhat more generally, namely for all 2-positive maps:

Theorem 1. *Let $\mathscr{H}$ and $\mathscr{H}'$ be two Hilbert spaces. For any $p, q \geqslant 1$, and any 2-positive linear map Φ acting from the Schatten classes of Hermitian operators $\ell_p(\mathscr{H})$ to $\ell_q(\mathscr{H}')$,*

$$\|\Phi\|_{p\to q} = \sup_{A} \frac{\|\Phi(A)\|_q}{\|A\|_p},$$

where A ranges over the Schatten p class of not necessarily Hermitian operators over $\mathscr{H}$.

Proof. Our proof uses the well-known construction of turning a general operator A on a Hilbert space $\mathscr{H}$ into a Hermitian one by defining an operator on the doubled space $\mathscr{H} \oplus \mathscr{H}$

$$Q := \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix},$$

which is obviously Hermitian.

There exists a unitary matrix U such that $(AA^*)^{1/2} = U|A|U^*$. The absolute value of Q is therefore

$$\begin{aligned} |Q| &= \left[\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}\right]^{1/2} \\ &= \begin{pmatrix} AA^* & 0 \\ 0 & A^*A \end{pmatrix}^{1/2} = \begin{pmatrix} U|A|U^* & 0 \\ 0 & |A| \end{pmatrix}. \end{aligned}$$

The doubled space $\mathscr{H} \oplus \mathscr{H}$ is isomorphic to the space $\mathscr{H} \otimes \mathbb{C}^2$. Let $\Phi \otimes \mathbb{1}_2$ be the extension map of Φ that operates trivially on $\mathbb{C}^2$. Then, using the fact that for any positive map $\Phi(A^*) = \Phi(A)^*$ (see, e.g. [7], but note that an easier proof is possible for 2-positive maps), we have

$$(\Phi \otimes \mathbb{1}_2)(Q) = \begin{pmatrix} 0 & \Phi(A) \\ \Phi(A)^* & 0 \end{pmatrix}.$$

Thus

$$
\begin{aligned}
\|(\Phi\otimes\mathbb{1}_2)(Q)\|_q^q &= \mathrm{Tr}|(\Phi\otimes\mathbb{1}_2)(Q)|^q \\
&= \mathrm{Tr}\left|\begin{pmatrix} 0 & \Phi(A) \\ \Phi(A)^* & 0 \end{pmatrix}\right|^q \\
&= \mathrm{Tr}\begin{pmatrix} (\Phi(A)\Phi(A)^*)^{1/2} & 0 \\ 0 & (\Phi(A)^*\Phi(A))^{1/2} \end{pmatrix}^q \\
&= 2\mathrm{Tr}|\Phi(A)|^q,
\end{aligned}
$$

where in the last line we have again used unitary equivalence of XX^* and X^*X.

Likewise,

$$
\begin{aligned}
\|(\Phi\otimes\mathbb{1}_2)(|Q|)\|_q^q &= \mathrm{Tr}(\Phi\otimes\mathbb{1}_2)(|Q|)^q \\
&= \mathrm{Tr}\begin{pmatrix} \Phi(U|A|U^*) & 0 \\ 0 & \Phi(|A|) \end{pmatrix}^q \\
&= \mathrm{Tr}\Phi(U|A|U^*)^q + \mathrm{Tr}\Phi(|A|)^q.
\end{aligned}
$$

Here we have used 2-positivity of Φ, implying positivity of $(\Phi\otimes\mathbb{1}_2)(|Q|)$.

Applying the inequality (2) to the map $\Omega=\Phi\otimes\mathbb{1}_2$ and the operator $X=Q$, and combining with the above results gives

$$2\mathrm{Tr}|\Phi(A)|^q \leqslant \mathrm{Tr}\Phi(U|A|U^*)^q + \mathrm{Tr}\Phi(|A|)^q.$$

Let us now impose the requirement $\|A\|_p=1$. Thus, obviously, both $|A|$ and $U|A|U^*$ have p-norm equal to 1. Then, by definition of the $p\to q$ norm of Φ,

$$\mathrm{Tr}\Phi(U|A|U^*)^q \leqslant \|\Phi\|_{p\to q}^q \quad\text{and}\quad \mathrm{Tr}\Phi(|A|)^q \leqslant \|\Phi\|_{p\to q}^q,$$

so that also

$$2\mathrm{Tr}|\Phi(A)|^q \leqslant 2\|\Phi\|_{p\to q}^q.$$

For general A, $A/\|A\|_p$ has p-norm equal to 1, so that by homogeneity,

$$\frac{\|\Phi(A)\|_q}{\|A\|_p} \leqslant \|\Phi\|_{p\to q}.$$

Maximising over all operators A, which includes the Hermitian ones, we find

$$\sup_A \frac{\|\Phi(A)\|_q}{\|A\|_p} = \|\Phi\|_{p\to q}.$$

And that finishes the proof. $\square$

Acknowledgement

This work was supported by The Leverhulme Trust (Grant F/07 058/U), and is part of the QIP-IRC (www.qipirc.org) supported by EPSRC (GR/S82176/0). I thank M.-B. Ruskai for urging me to publish this note.

References

[1] G.G. Amosov, A.S. Holevo, On the multiplicativity hypothesis for quantum communication channels,Theory Prob. Appl. 47 (2002) 123–127.
[2] K.M.R. Audenaert, On the Araki–Lieb–Thirring inequality, Int. J. Inform. Syst. Sci. 4 (2008) 78–83.
[3] M.-D. Choi, Completely positive linear maps on complex matrices, Linear Algebra Appl. 10 (1975) 285–290.
[4] C. King, M.B. Ruskai, Comments on multiplicativity of maximal p-norms when $p=2$, Quant. Inform. Comput. 4 (2004) 500–512.

[5] K. Kraus, States, Effects, and Operations, Springer-Verlag, Berlin, 1983.
[6] M.A. Nielsen, I.L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, 2000.
[7] R. Schrader, Perron–Frobenius theory for positive maps on trace ideals, in: Roberto Longo (Ed.), Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects, Fields Inst. Commun., vol. 30, Amer. Math. Soc., Providence, RI, 2001 (see also arXiv eprint math-ph/0007020).
[8] J. Watrous, Notes on super-operator norms induced by Schatten norms, Quant. Inform. Comput. 5 (2005) 58–68.